Output-Feedback Control of Nonlinear Plants

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For multiinput-multioutput nonlinear plants whose state-feedback control problem is solvable with complete or partial output linearization, the output-feedback problem is addressed by combining the state-feedback controller with a suitable closed-loop state inferer (detector). A candidate closed-loop detector was built to study the stability of the resulting plant-controller interconnection. As a result, sufficient conditions for closed-loop asymptotic nominal stability, as well as a systematic and simple design-tuning procedure, are obtained. The stabilization of an open-loop, unstable, free-radical homopolymerization reactor was studied as an application example.

Introduction

In the last ten years there has been an increasing interest in the treatment of process control problems with nonlinear geometric control techniques as a topic of research in chemical engineering (Bequette, 1991). This research is an effort to search for a nonlinear model-based control framework that can be a match for the existing and successful open-loop (i.e., without control) nonlinear dynamics approaches to modeling and its applications to process analysis and design. A diversity of chemical process control systems have been studied with the following nonlinear geometric techniques: state-feedback control (Hoo and Kantor, 1985a; Kravaris and Chung, 1987; Alvarez et al., 1989); nonlinear state observation (i.e., inference of the states from the measured output) (Kantor, 1989); state-feedback control with input saturation (Alvarez et al., 1991); and output-feedback control (Hoo and Kantor, 1985b; Kantor, 1989; Kravaris and Chung, 1987; Henson and Seaborg, 1991; Soroush and Kravaris, 1993; Daoutidis and Kravaris, 1994). In particular, Soroush and Kravaris' experimental work demonstrates the practical applicability of the geometric nonlinear control. Until recently, geometric output-feedback controllers in chemical engineering have been designed by combining theoretically backed state-feedback controllers with *ad hoc* state estimators (i.e., observers and detectors), and consequently the corresponding output-feedback controllers have lacked closed-loop stability criteria and systematic design-tuning procedures.

At present, the geometric nonlinear control theory provides results on the state-feedback problem (Isidori, 1989; Nijmeijer and Van der Shaft, 1990) that apply to an important class of chemical control processes, and results on the observation problem (Krener and Isidori, 1983) that apply to a very restricted class of processes. Outside the geometric theory, there are three main approaches to the state-inference problem: The extended Kalman filter (EKF) (Meditch, 1969), whose design is simple but lacks stability criteria and systematic tuning procedures; the high-gain (HG) approach (Gauthier et al., 1992; Deza et al., 1992a; Ciccarella et al., 1993), which guarantees stability but has complex tuning procedures; and the sliding-mode (SM) approach (Slotine et al., 1987; Walcott et al., 1987) which guarantees robust stability but has an elaborated design. These three state-inference techniques are restricted to nonlinear plants that are completely observable in a suitable open-loop nonlinear sense (Krener and Respondek, 1985), and their extensions to partially observable (i.e., detectable) plants or plants in closed-loop operation do not seem straightforward, as can be seen in the work of Tsinias (1990), where Lyapunov-like conditions are given for the existence of a nonlinear open-loop state-detector. In chemical engineering, the EKF is by far the most widely used technique (Dimitratos et al., 1991; Ellis et al., 1994; Baratti et al., 1983, 1995), while the HG and SM approaches are rarely used (Deza et al., 1992b, 1993).

In linear systems, the closed-loop stability of a plant subjected to the combination of a stabilizing state-feedback with a stable detector is guaranteed by the separation principle (Wonham, 1985), but the same is not true for nonlinear plants (Vidyasagar, 1980). Consequently, the closed-loop stability of a particular detector-based controller must be assessed regardless of the individual stabilities of its detector and state-feedback controller. Assuming the existence of a stabilizing state-feedback controller for a distillation column, Deza et al. (1992b) established that the combination of such a controller with their earlier HG observer design (Gauthier et al., 1992) produced a stable closed-loop distillation column. However, the technique is restricted to staged processes where the assumed state-feedback controller does not modify the open-loop observability of the plant, and cannot be applied to partially state-feedback linearizable plants, which represent the common case in chemical processes. Tornambe (1992) and Hammouri and Busawon (1993) designed output-feedback controllers for nonlinear plants that are linear in the control and whose state-feedback problem is solvable by a complete feedback linearization. In both cases, the resulting controllers contain high-gain-type closed-loop observers. For applied chemical process control, these designs have two drawbacks: the class of plants that admit complete state-feedback linearization is restrictive; and the design-tuning procedure is rather complex, and of little appeal to a process control practitioner.

In a nonlinear geometric control methodology for applied process control, the following issues should be addressed: nominal and robust (i.e., with tolerance to modeling errors) stability and performance, the handling of input saturation and constraints, and the case of a system having more control inputs than regulated outputs. In particular, an effort should be made to produce control designs that are transparent as well as appealing to the process control designer, meaning that the control constructions, stability conditions, and design-tuning procedures should be contextualized and interpreted beyond the mathematical-result format encountered in the theoretical control literature.

This work addresses the geometric output-feedback problem with emphasis on conditions for nominal stability, the interplay between the controller and observer designs, and a systematic and simple design-tuning procedure. The problem is addressed for nonlinear multiinput-multioutput (MIMO) plants whose state-feedback problem can be solved either with complete or partial linearization, encompassing a broad range of applied process control systems. First, a closed-loop detector is built from the structure of the solution of the state-feedback problem. Then, the stability of the resulting plant-controller interconnection is studied to obtain sufficient conditions for closed-loop stability, to identify the interplay between the controller and observer designs, and to develop a simple design-tuning procedure that can be handled and interpreted with notions and tools from conventional control techniques for linear systems. The control of an open-loop, unstable, continuous, free-radical homopolymerization reactor is addressed as an application example, focusing on solvability conditions with physical meaning, and resorting to numerical simulations to corroborate and illustrate the findings.

Class of Plants and Control Problem

Let us consider nonlinear MIMO plants

$$\dot{x} = f(x,u), \quad y = h(x); \quad x \in X_w, \quad u \in U_w \tag{1}$$

with n states (x), m control inputs (u) and m outputs (y). The maps f and h are smooth (infinitely differentiable) in the set $X_w \times U_w$, and the plant must operate at a (possibly open-loop unstable) nominal steady-state $\bar{x}$ with nominal input $\bar{u}$, and nominal output $\bar{y}$. This is,

$$f(\bar{x},\bar{u}) = 0, \quad \bar{y} = h(\bar{x}) \tag{2}$$

The geometric (static) state-feedback problem of the plant (Eq. 1) is assumed to be solvable, either with complete or partial state-feedback linearization (Isidori, 1989; Nijmeijer and Van der Shaft, 1990). This is, in some state neighborhood X (contained in X_w) of $\bar{x}$, there is a smooth nonlinear controller

$$u = \mu(x, K_c), \quad x \in X \tag{3}$$

that, applied to the plant, yields a closed-loop system

$$\dot{x} = f[x, \mu(x, K_c)], \quad y = h(x); \quad x \in X \tag{4}$$

where $\bar{x}$ is an asymptotically stable (A-stable) steady state, K_c is a block-diagonal control gain matrix (see Appendix A) with κ adjustable entries, which are the coefficients of the linear, noninteractive and pole-assignable (LNPA) closed-loop output dynamics

$$\eta_i^{(\kappa_i)} - k_{i\kappa_i}^c \eta_i^{(\kappa_i - 1)} - \ldots - k_{i1}^c \eta_i = 0; \quad \eta_i = y_i - \bar{y}_i,$$
$$\kappa_1 + \ldots + \kappa_m = \kappa \leq n, \quad 1 \leq i \leq m \tag{5}$$

of the closed-loop system (Eq. 4). The dynamics of the ith output is independent of the other outputs, and the entries of K_c can be tuned to shape the stable responses of all the outputs. Regarding applied process control, this LNPA feature is important because it enables a tuning procedure based on any of the available conventional techniques for linear single input–single output (SISO) systems.

The solution of the state-feedback problem is based on the assumption that the state x of the plant is measured and available for control. However, in a practical situation only the measured output y is available for control, and therefore, the state-feedback controller must be driven by an inferred value χ of the state, produced by a suitable on-line closed-loop detector of form (Eq. 6a), where $G(\chi, K_o)$ is a nonlinear matrix map and K_o is an adjustable gain matrix. If such a detector exists, its combination with the state-feedback (Eq. 3) controller yields the following dynamic output-feedback controller:

$$\dot{\chi} = f[\chi, \mu(\chi, K_c)] + G(\chi, K_o)[y - h(\chi)] \tag{6a}$$

$$u = \mu(\chi, K_c) \tag{6b}$$

Having as a point of departure the solvability of the state-feedback problem of the plant (Eq. 1) and the construction of its related feedback map $\mu(\chi, K_c)$, knowing that the separation principle does not apply *a priori* to any nonlinear plant, and with the premise that the simplicity of the LNPA design of the state-feedback problem must be carried over as much as possible to the design of the output-feedback controller, three main issues are addressed in this work: (i) the construction of the detector nonlinear gain $G(\chi, K_o)$; (ii) the nominal stability of the resulting plant-controller interconnection; and (iii) the obtainment of a gain tuning procedure.

Construction of Candidate Output-Feedback Controller

Similar to the derivation of a detector-based controller from the separation principle for linear control systems (Wonham, 1985), a candidate output-feedback controller is built as follows: (i) assuming that the state of the plant is known, build a state-feedback controller; (ii) assuming that the plant is subjected to an input-output linearizing controller, build a closed-loop detector; and (iii) consider the combination of the controller with the detector as a candidate output-feedback controller.

State-feedback control

For the purpose at hand, in this section we recall the solvability conditions of the state-feedback problem (Isidori, 1989; Nijmeijer and Van der Shaft, 1990) in a format that highlights the properties of interest: state-stability and LNPA output dynamics, but introducing one modification: instead of recalling the usual local necessary and sufficient solvability conditions, nonlocal (i.e., in a given state-input set $X_w \times U_w$) sufficient conditions will be presented. By doing so, we will be able to study (in the next section) the stability of the closed-loop plant in some neighborhood $X \in X_w$, of the nominal steady-state $\bar{x}$ without worrying about having singularities in the feedback map μ of the controller (Eq. 3).

Definition 1. The (static) state-feedback problem of the plant (Eq. 1) is solvable if there exists a smooth nonlinear controller (Eq. 3) such that, in some neighborhood $X \subset X_w$ of the (possibly open-loop unstable) nominal steady-state $\bar{x}$, the resulting closed-loop system (Eq. 4) has:

(i) $\bar{x}$ as an A-stable steady state;

(ii) LNPA output dynamics (Eq. 5). ♦

To state the solvability conditions of this problem, the definition of the directional derivative of the scalar field $c(x)$ with respect to the vector field $f(x)$, and of its recursion formula must be introduced

$$L_f^{i+1}c = L_f\left(L_f^i c\right), \quad L_f^o c = c, \quad L_f c = (\partial c/\partial x_1) f_1 + \ldots + (\partial c/\partial x_n) f_n$$

Theorem 1. The state-feedback problem of the plant (Eq. 1) is solvable if there are m strictly positive integers (output relative degrees), $\kappa_1, \ldots, \kappa_m$, and $n-\kappa$ scalar maps $\phi_{\kappa+1}(x), \ldots, \phi_m(x)$, such that, in a given state-input set $X_w \times U_w$:

(i) The relative degrees meet the following condition

$$\kappa_1 + \ldots + \kappa_m = \kappa \leq n, \quad \kappa_i > 0$$

(ii) The map ϕ_I is independent of u

$$\phi_I(x) = \left[h_1, \ldots, L_f^{\kappa_1 - 1}h_1; \ldots; h_m, \ldots, L_f^{\kappa_m - 1}h_m\right]^T$$

(iii) The map ϕ is invertible: $\phi^{-1}[\phi(x)] = x$,

$$\phi(x) = \begin{bmatrix} \phi_I(x) \\ \phi_{II}(x) \end{bmatrix}, \phi_{II}(x) = [\phi_{\kappa+1}(x), \ldots, \phi_m(x)]^T$$

(iv) The map φ is invertible: $\varphi^{-1}[x, \varphi(x,u)] = u$,

$$\varphi(x,u) = [L_f^{\kappa_1}h_1, \ldots, L_f^{\kappa_m}h_m]^T$$

(v) $\bar{x}$ is a unique and exponentially (E) stable steady state of the $(n-\kappa)$-dimensional zero-dynamics

$$\dot{x} = f[x, \varphi^{-1}(x,0)], \quad x \in \{x \in X_w \mid \phi_I(x) = \phi_I(\bar{x})\} \subset X_w, \quad \dim X_z = n - \kappa. \blacklozenge \quad (7)$$

The proof of this theorem is given in Appendix B, which is included because it contains material required in the subsequent construction of the detector. If the conditions of the above theorem are met, the nonlinear controller (Eq. 3) is given by (derivation in Appendix B)

$$u = \varphi^{-1}\{x, K_c[\phi_I(x) - \phi_I(\bar{x})]\}, \quad K_c \in \mathcal{K}_c \qquad (8)$$

where $\mathcal{K}_c$ (see Appendix A) denotes the set of control gains K_c that make stable the LNPA output dynamics (Eq. 5).

Closed-loop state-detection

Let us regard the state-feedback controller (Eq. 3) as the combination of two state-feedback controllers, one

$$u = \varphi^{-1}(x, \upsilon)$$

that accomplishes input (υ)-output(y) linearization, and one

$$\upsilon = K_c[\phi_I(x) - \phi_I(\bar{x})]$$

that sets the LNPA output dynamics. Let us assume that the plant (Eq. 1) is subjected to the former controller with a known, exogenous, bounded, and asymptotically vanishing input $\upsilon(t)$: $\|\upsilon(t)\| < \infty$, $\upsilon(t) \to 0$ as $t \to \infty$. The resulting auxiliary closed-loop system is

$$\dot{x} = f\{x, \varphi^{-1}[x, \upsilon(t)]\}, \quad y = h(x) \qquad (9)$$

From the invertibility property of the map $\phi(x)$ of theorem 1, the coordinate change

$$z = \phi(x) - \phi(\bar{x}), \quad \eta = y - h(\bar{x}) \qquad (10)$$

takes the auxiliary closed-loop system (Eq. 9) into a detectable normal form (Eq. B2 of Appendix B), which is a cascade interconnection of two subsystems: one (Eq. B2a) that is linear, decoupled, and completely observable; and one (Eq. B2b) that is nonlinear, unobservable, and driven by the state of the former subsystem. From these properties and the E-stability of the zero-dynamics (Eq. 7 or Eq. B2b with $z_I = 0$), it follows that the system

$$\dot{\zeta}_I = \Gamma\zeta_I + K_o(\eta - \Delta\zeta_I), \quad K_o \in \mathcal{K}_o \qquad (11a)$$

$$\dot{\zeta}_{II} = w\{\zeta_I, \zeta_{II}, \varphi^{-1}[\alpha(\zeta_I, \zeta_{II}), \upsilon(t)]\} \qquad (11b)$$

is an A-detector for the normal form (Eq. B2), or equivalently, for the auxiliary closed-loop system (Eq. 9) in z-coordinates. K_o has a block-diagonal structure, inherited from the state-feedback problem, and κ adjustable entries, which are the coefficients of the following LNPA output error dynamics

$$v_i^{(\kappa_i)} + k_{1i}^o v_i^{(\kappa_i - 1)} + \ldots + k_{\kappa_i i}^o v_i = 0, \quad v_i = h_i(\chi) - y_i, \quad 1 \le i \le m \quad (12)$$

where $\mathcal{K}_o$ (see Appendix A) is the set of observer gains K_o that make stable this output dynamics. The existence of $\mathcal{K}_o$ is guaranteed by the observability of the matrix pair (Γ, Δ) of the linear subsystem (Eq. B2a) of the normal form (Eq. B2). To obtain the preceding output dynamics, subtract Eq. B2a from Eq. 11a, and take the κ_i time-derivative of the ith output map. In original coordinates, the detector (Eq. 11) is given by

$$\dot{\chi} = f\{\chi, \varphi^{-1}[\chi, v(t)]\} + G(\chi, K_o)[y - h(\chi)], \quad (13a)$$

$$G(\chi, K_o) = \Omega(\chi) K_o, \quad [\phi_x^{-1}(\chi)] = [\Omega(\chi), \Xi(\chi)] \quad (13b)$$

where the matrix $\Omega(\chi)$ corresponds to the first κ columns of the inverse of the Jacobian matrix ϕ_x of the map $\phi(x)$, defined in Theorem 1. If the plant is completely linearizable (i.e., $\kappa = n$), $G(\chi) = \phi_x^{-1}(\chi) K_o$, and $\phi_x(x)$ is the so-called observability matrix that appears in one of the solvability conditions of the geometric nonlinear observation problem (Krener and Isidori, 1983), and appears as well in some open-loop observer designs for plants that are completely observable (Bestle and Zeitz, 1983; Ciccarella et al., 1993).

Candidate output-feedback controller

In the last two subsections, two unrealizable problems have been solved separately: the state-feedback problem (assuming the state is known), and the closed-loop state-detection problem (assuming the plant is subjected to state-feedback output-linearizing control). The combination of the corresponding controller (Eq. 8) and detector (Eq. 13) yields the following candidate for output-feedback controller

$$\dot{\chi} = f[\chi, \mu(\chi; K_c)] + G(\chi; K_o)[y - h(\chi)], \quad K_o \in \mathcal{K}_o \quad (14a)$$

$$u = \mu(\chi; K_c), \quad K_c \in \mathcal{K}_c \quad (14b)$$

By candidate for controller, we mean that its validity will be established *a posteriori* (in the next section), after studying its closed-loop dynamics and concluding on its stability.

Closed-Loop Stability

In this section, sufficient conditions for the closed-loop stability of the interconnection of the plant with the candidate for dynamic output-feedback controller are derived, and a procedure for gain tuning is given.

The application of the candidate for controller (Eq. 14) to the plant (Eq. 1) yields the following closed-loop system

$$\dot{x} = f[x, \mu(\chi; K_c)], \quad x \in X \quad (15a)$$

$$\dot{\chi} = f[\chi, \mu(\chi; K_c)] + G(\chi; K_o)[h(x) - h(\chi)], \quad \chi \in X \quad (15b)$$

From Eq. 2 and $\bar{u} = \mu(\bar{x}; K_c)$ follows that $\bar{x}_c = [\bar{x}^T, \bar{\chi}^T]^T$ is a steady state of the closed-loop plant. Hence, the question is as to how the 2κ gains of the controller-observer gain pair (K_c, K_o) can be chosen to ensure that the nominal steady state $\bar{x}_c$ of the preceding closed-loop system is A-stable.

Formulation of stability problem

In a way analogous to the proof of the separation principle in linear systems, let us introduce the coordinate change

$$e = \phi(x) - \phi(\chi), \quad \zeta = \phi(\chi) - \phi(\bar{x}) \quad (16)$$

to take the closed-loop plant (Eq. 15) into the form

$$\dot{e}_{\rm I} = A_o e_{\rm I} + \Pi q_{\rm I}(e_{\rm I}, e_{\rm II}, \xi_{\rm I}, \xi_{\rm II}) \quad (17a)$$

$$\dot{\zeta}_{\rm I} = A_c \zeta_{\rm I} - K_o \Delta e_{\rm I}, e \in E, \zeta \in \mathcal{Z} \quad (17b)$$

$$\dot{\zeta}_{\rm II} = w[\zeta_{\rm I}, \zeta_{\rm II}, \nu(\zeta_{\rm I}, \zeta_{\rm II})] \quad (17c)$$

$$\dot{e}_{\rm II} = w^*(e_{\rm I}, e_{\rm II}, \zeta_{\rm I}, \zeta_{\rm II}) \quad (17d)$$

where the maps $q_{\rm I}$, w, and w^* are defined in Appendix A. $E \times \mathcal{Z}$ is in (e, ζ)-coordinates the neighborhood $X \times \mathfrak{X}$ of the closed-loop system (Eq. 15). Equations 17a and 17d describe the detection error dynamics of the feedback-linearizable (controllable-observable) and the nonlinearizable (unobservable) parts of the plant, respectively, and Eqs. 17b and 17c describe the detector dynamics associated with the same parts of the plant. If $q_{\rm I} = 0$, the preceding system is A-stable because it consists of a cascade interconnection of four subsystems that are individually stable (Vidyasagar, 1978). In original coordinates, this key error feedback is given by

$$q_{\rm I} = \varphi\{\chi, K_c[\phi(\chi) - \phi(\bar{x})]\} - \varphi\{x, K_c[\phi(\chi) - \phi(\bar{x})]\}$$

As mentioned at the beginning of the subsection on state-feedback control $\varphi(x, v)$ is an input-output linearizing map, and therefore, the preceding equation shows why the separation principle breaks down when the controller (Eq. 8) and the detector (Eq. 13) are combined: because $\varphi(\chi, v) \neq \varphi(x, v)$ or, equivalently, because of the impossibility of achieving exact input-output linearization with any detector-based controller.

Let us rewrite the preceding closed-loop dynamics as follows

$$\dot{e}_{\rm I} = A_o e_{\rm I} + \Pi q_{\rm I}(e_{\rm I}, e_{\rm II}, \zeta_{\rm I}, \zeta_{\rm II}), \quad q_{\rm I}(0, 0, \zeta_{\rm I}, \zeta_{\rm II}) = 0 \quad (18a)$$

$$\dot{\zeta}_{\rm I} = A_c \zeta_{\rm I} + \theta_{\rm I}(e_{\rm I}), \quad \theta_{\rm I}(0) = 0 \quad (18b)$$

$$\dot{\zeta}_{\rm II} = \omega(\zeta_{\rm II}) + \theta_{\rm II}(\zeta_{\rm I}, \zeta_{\rm II}), \quad \theta_{\rm II}(0, \zeta_{\rm II}) \quad (18c)$$

$$\dot{e}_{\rm II} = \overline{\omega}(e_{\rm II}) + q_{\rm II}(e_{\rm I}, e_{\rm II}, \zeta_{\rm I}, \zeta_{\rm II}), \quad q_{\rm II}(0, e_{\rm II}, 0, 0) = 0 \quad (18d)$$

with the maps q_{II}, θ_I, θ_{II}, ω, and $\bar{\omega}$ defined in Appendix A. If we regard the functions q_I, q_{II}, θ_I and θ_{II} as error-feedbacks that are all zero, the resulting closed-loop dynamics are stable. Thus, the question is whether the stability property of such "basic dynamics" (i.e., Eq. 18 without error feedbacks) can withstand the presence of the nonlinear "error feedbacks" q_I, q_{II}, θ_I and θ_{II}. Next we proceed to characterize the stability properties of the closed-loop basic dynamics, and the "sizes" of their corresponding error feedbacks.

In virtue of the stability of the controller and observer matrices A_o and A_c, there are constants, namely a_o, a_c (amplitude factors), λ_o, λ_c (decay factors), that bound the exponential responses of those matrices

$$\|\exp(A_o t)\| \le a_o \exp(-\lambda_o t), \quad \|\exp(A_c t)\| \le a_c \exp(-\lambda_c t) \tag{19}$$

In z-coordinates (Eq. 10), the E-stable zero-dynamics (Eq. 7) becomes

$$\dot{z}_{II} = \omega(z_{II}), \quad z_I = 0, \quad z_{II} \in Z_{II} \tag{20}$$

and its motions are bounded as follows

$$\|z_{II}(t)\| \le a_z \|z_{IIo}\| \exp(-\lambda_z t) \tag{21}$$

where a_z and λ_z are the amplitude and decay factors, respectively, of the zero-dynamics. From this inequality and given Lyapunov's converse theorem (Khalil, 1992), there is a Lyapunov function $V(z_{II})$ and two positive constants c_1 and b_z such that along the motion $z_{II}(t)$ of the system (Eq. 20) the following inequality (In. C6) holds

$$\|(\partial V/\partial z_{II})\| \le -c_1 b_z \tag{22}$$

From the smoothness of maps f and h of the plant (Eq. 1), the error-feedbacks of the closed-system (Eq. 18) are in some neighborhood $E \times \mathcal{Z}$ bounded (Vidyasagar, 1978) as follows

$$\|q_I(e_I, e_{II}, \zeta_I, \zeta_{II})\| \le L_1(E, \mathcal{Z}, K_c)\|e_I\| + L_2(E, \mathcal{Z}, K_c)\|e_{II}\| \tag{23a}$$

$$\|\theta_I(e_I)\| \le k_o(K_o)\|e_I\|, \quad k_o(K_o) = \|K_o\| \tag{23b}$$

$$\|\theta_{II}(\zeta_I, \zeta_{II})\| \le L_3(\mathcal{Z}, K_c)\|\zeta_I\| \tag{23c}$$

$$\|q_{II}(e_I, e_{II}, \zeta_I, \zeta_{II})\| \le L_4(E, \mathcal{Z}, K_c)\|e_I\| + L_5(E, \mathcal{Z}, K_c)\|\zeta_I\| + L_6(E, \mathcal{Z}, K_c)\|\zeta_{II}\| \tag{23d}$$

$$L_i \to \ell_i (1 \le i \le 6), \ \ell_5 = \ell_4, \quad \text{as } E \times \mathcal{Z} \to 0$$

with k_o and L_i being Lipschitz bounds, and ℓ_1 their limits. $k_o(K_o)$, and $\ell_3(K_c)$ are gain dependent, and the other limits are gain independent.

Summarizing, the inequality sets (Eqs. 19, 21, and 22) characterize the stability properties of control-observer LNPA design, and of the zero-dynamics, and the inequality set (Eq. 23) characterizes the sizes (k_o, ℓ_1, ..., ℓ_6) of the error-feedback terms (q_I, q_{II}, θ_I, θ_{II}) of the basic closed-loop dynamics (Eq. 18). Thus, our closed-loop stability problem amounts to the following: find conditions on the admissible controller and observer gains $K_c \in \mathcal{K}_c$ and $K_o \in \mathcal{K}_o$ such that the potential destabilization due to the error-feedback functions of the system (Eq. 18) can be dominated.

Closed-loop stability for completely feedback-linearizable plants

When the state-feedback problem of the plant is solvable with complete linearization $\kappa = n$, $e = e_I$, $\zeta = \zeta_I$, there is no zero-dynamics and the closed-loop dynamics (Eq. 18) is given by

$$\dot{e} = A_o e + \Pi q_I(e, \zeta), \quad v = \Delta e \tag{24a}$$

$$\dot{\zeta} = A_c \zeta - K_o \Delta e, \quad \eta = \Delta(\zeta - e) \tag{24b}$$

Without the error feedback q_I, the preceding system coincides with the one obtained from the application of the separation principle to the output-feedback control problem of the controllable-observable normal form (Eq. B2a). In the next lemma (proof in Appendix C), a sufficient condition for the closed-loop stability of the plant is given.

Lemma 1a. Let the state-feedback problem of the plant (Eq. 1) be solvable with complete linearization ($\kappa = n$). Then, the output-feedback controller (Eq. 14) yields an A-stable closed-loop system (Eq. 15) if

i) $\dfrac{\lambda_o(K_o)}{a_o(K_o)} > \ell_1, \quad K_o \in \mathcal{K}_o.$ ◆

The fulfillment of this condition can be seen as meeting the specification of a minimum stability margin ($\lambda_o - a_o \ell_1$). This has the structure of the high-gain approach for the design of nonlinear observers (Gauthier et al., 1992; Ciccarella et al., 1993), and resembles the preclusion of the so-called peaking phenomenon in the state-feedback stabilization of a nonlinear plant (Sussmann and Kokotovic, 1991).

The closed-loop output dynamics reaches asymptotically the design LNPA output dynamics. In other words, this design dynamics determines to a good extent the actual closed-loop output dynamics. With respect to applied process control, this means that the closed-loop output response can be shaped by tuning the design LNPA output dynamics, using standard design techniques for linear SISO control systems. As we shall see in the next subsection, with some additional considerations, the same conclusion holds for the case of a partially linearizable plant.

Closed-loop stability for partially feedback-linearizable plants

In the case of a partially linearizable plant, its closed-loop dynamics is more complex, due to the presence of the zero-dynamics and its coupling with the linearizable part of the plant. Accordingly, the derivation of the stability conditions is more elaborated, as can be seen in Appendix C, where the following lemma is proved.

Lemma 1b. Let the state-feedback problem of the plant (Eq. 1) be solvable with partial linearization ($\kappa < n$). The closed-loop plant (Eq. 15) is A-stable if

i) $\dfrac{\lambda_o(K_o)}{a_o(K_o)} > \ell_1, \quad K_o \in \mathcal{K}_o, \quad K_c \in \mathcal{K}_c$

ii) $\dfrac{\lambda_o(K_o)}{a_o(K_o)} \neq \ell_1 + \ell_2 \alpha_z \left\{ \ell_4 \left(1 + \dfrac{a_c(K_c)}{\lambda_c(K_c)} k_o(K_o) \right) + \ell_6\, \ell_3(K_c) \alpha_z \right\}.$

where $\alpha_z = \lim (b_2 a_2/\lambda_2)$ as $E \times z \to 0$. ♦ Condition (i) is the same condition as that of Lemma 1b, and condition (ii) is a nonresonance-like condition to prevent a potential destabilization due to the presence of the unobservable zero-dynamics.

Stability with parametrized output-feedback controller

According to the preceding lemmas, the key to meet their stability conditions resides in being able to tune the controller and observer gains K_c and K_o, such that their decay-to-amplitude ratios, λ_c/a_c and λ_o/a_o, can be varied arbitrarily and independently. For this purpose, in this section we introduce a root locus-type approach to the tuning of the control-observer gains.

Let us consider the LNPA control (Eq. 5) and the observer (Eq. 12) output dynamics with prespecified reference admissible gains $K_{cr} \in \mathcal{K}_c$ and $K_{or} \in \mathcal{K}_o$ and introduce two time-scaling parameters s_c and s_o for the control and observer designs, respectively

$$\frac{d^{\kappa_i}\eta_i}{d(s_c t)^{\kappa_i}} - k^{cr}_{i\kappa_i}\frac{d^{(\kappa_i-1)}\eta_i}{d(s_c t)^{(\kappa_i-1)}} - \ldots - k^{cr}_{i1}\eta_i = 0, \quad s_c > 0, \quad 1 \le i \le m$$

$$\frac{d^{\kappa_i}v_i}{d(s_o t)^{\kappa_i}} + k^{or}_{1i}\frac{d^{(\kappa_i-1)}v_i}{d(s_o t)^{(\kappa_i-1)}} + \ldots + k^{or}_{\kappa_i i}v_i = 0, \quad s_o > 0$$

Observe that the coefficients of these equations are the entries of K_{cr} and K_{or}. For the output dynamics of the controller (Eq. 5) and the observer (Eq. 12) designs to coincide with the last parametrized equations, the coefficients of Eqs. 5 and 12 must be parametrized in the following manner

$$k^c_i(k^{cr}_i, s_c) = [s_c^{\kappa_i}k^{cr}_{i1}, \ldots, s_c k^{cr}_{i\kappa_i}], \quad s_c > 0;$$

$$k^o_i(k^{or}_i, s_o) = [s_o k^{or}_{1i}, \ldots, s_o^{\kappa_i}k^{or}_{\kappa_i i}]^T, \quad s_o > 0$$

Accordingly, the gain matrices K_c and K_o their related matrices A_c and A_o, and the inequality set (Eq. 19) take the parametrized forms

$$K_c(K_{cr}, s_c) = bd[k^c_1(k^{cr}_1, s_c), \ldots, k^c_m(k^{cr}_m, s_c)], \quad s_c > 0, \quad K_{cr} \in \mathcal{K}_c \quad (25a)$$

$$K_o(K_{or}, s_o) = bd[k^o_1(k^{or}_1, s_o), \ldots, k^o_m(k^{or}_m, s_o)], \quad s_o > 0, \quad K_{or} \in \mathcal{K}_o \quad (25b)$$

$$A_c(K_{cr}, s_c) = \Gamma + \Pi K_c(K_{cr}, s_c), \quad A_o(K_{or}, s_o) = \Gamma - [K_o(K_{or}, s_o)]\Delta$$

$$\|\exp[A_c(s_c)t]\| \le a_{cr}\exp(-s_c\lambda_{cr}t), \quad \|\exp[A_o(s_o)t]\| \le a_{or}\exp(-s_o\lambda_{or}t)$$

The comparison of the last inequality set with the inequality set (Eq. 19) yields that

$$\lambda_c(s_c) = s_c\lambda_{cr}, \quad a_c = a_{cr}; \qquad \lambda_o(s_o) = s_o\lambda_{or}, \quad a_o = a_{cr}$$

Thus, the parametrized decay factor $\lambda_{c/o}(s_{c/o})$ depends linearly on the acceleration (or retardation) parameter $s_{c/o}$, and the amplitude factor $a_{c/o}$ is independent of $s_{c/o}$, or equivalently, the parametrized decay-to-amplitude quotient $[\lambda_{c/o}(s_{c/o})]/a_{c/o}$ can be varied at will. This implies that the two stability inequalities of Lemmas 1a and 1b can be always met by a suitable choice of the parameter pair (s_c, s_o). This conclusion is presented in the next theorem, where the parametrized version of the stability conditions of Lemmas 1a and 1b are given, and the main construction steps and design-tuning tools required for implementation are included.

Theorem 2. Let the state-feedback problem of the (n-state, m-input, m-output) plant (Eq. 1) be solvable with relative degrees $\kappa_1, \ldots, \kappa_m$ $(\kappa_i + \ldots + \kappa_m)$, let

$$\eta_i^{(\kappa_i)} - k^{cr}_{i\kappa_i}\eta_i^{(\kappa_i-1)} - \ldots - k^{cr}_{i1}\eta_i = 0 \quad (26a)$$

$$v_i^{(\kappa_i)} + k^{or}_{1i}v_i^{(\kappa_i-1)} + \ldots + k^{or}_{\kappa_i i}v_i = 0, \quad (26b)$$

be the reference controller and observer (stable) LNPA output dynamics, let

$$K_{cr} = bd[k^{cr}_1, \ldots, k^{cr}_m] \in \mathcal{K}_c, \quad k^{cr}_i = \left[k^{cr}_{i1}, \ldots, k^{cr}_{i\kappa_i}\right]$$

$$K_{or} = bd[k^{or}_1, \ldots, k^{or}_m] \in \mathcal{K}_o, \quad k^{or}_i = [k^{or}_{1i}, \ldots, k^{cr}_{\kappa_i i}]^T$$

be their corresponding reference control and observer gains, and let the controller and observer gains be in the parametrized forms Eq. 25a, and Eq. 25b, respectively. Then, the parametrized dynamic output-feedback controller

$$\dot{\chi} = f\{\chi, \mu[\chi, K_c(K_{cr}, s_c)]\} + G[\chi, K_o(K_{or}, s_o)][y - h(\chi)] \quad (27a)$$

$$u = \mu[\chi, K_c(K_{cr}, s_c)] \quad (27b)$$

yields a closed-loop system Eq. 15 if

(i) The stability margin condition, if $\kappa \le n$

$$s_o > \alpha^r_o(K_{or})\,\ell_1$$

(ii) Nonresonance condition, if $\kappa < n$

$$s_o \neq \alpha^r_o(K_{or})\{\ell_1 + \ell_2\alpha_z\{\ell_4[1 + k_o(K_{or}, s_o)\alpha^r_c(K_{cr})/s_c] + \alpha_z\,\ell_6\,\ell^r_3(K_{cr}, s_c)\}\}$$

where

$$\alpha_c^r = a_{cr}/\lambda_{cr}, \quad \alpha_o^r = a_o/\lambda_{or}, \quad \alpha_z = \lim(b_z a_z/\lambda_z) \text{ as } E \times Z \to 0$$

$$k_o(K_{or}, s_o) = \|K_o(K_{or}, s_o)\|, \; \ell_3^r(K_{cr}, s_c) = \ell_3[K_c(K_{cr}, s_c)]. \blacklozenge$$

If the plant is completely linearizable (i.e., $\kappa = n$), the stability of the closed-loop plant can be established by fulfilling just condition (i), which can be done by choosing the observer parameter (s_o) sufficiently large, or equivalently, by making the observer design sufficiently fast. If the plant is partially linearized (i.e., $\kappa < n$), the two stability conditions of the last theorem can be met by choosing the observer (s_o) and control (s_c), parameters sufficiently large and small, respectively; or equivalently, by making the observer and control designs sufficiently fast or slow, respectively. The two conditions show the interplay between the attainment of closed-loop stability, the stabilities of the controller-observer LNPA design, the stability of the zero-dynamics, the sizes of the error feedbacks, the size of the neighborhood of attraction, and the choice of parametrized controller-observer gains. The root-locus technique enables the setting and displacement of the poles of the controller and observer LNPA designs, guaranteeing closed-loop stability and enforcing some desired properties (settling time, damping, overshoot, rise time, attenuation of modeling and design errors, etc.) in the closed-loop output-dynamics. For this purpose, in addition to the tuning parameters s_o and s_c, the 2κ parameters of the controller and observer reference gains K_{cr} and K_{or} can be adjusted.

Concluding remarks

If the plant (Eq. 1) is SISO and completely linearizable (i.e., $\kappa = n$, $m = 1$), the parametrized closed-loop observer gain (Eq. 25 with $\kappa = n$, $m = 1$) coincides with the one of Deza et al. (1992a), which was obtained within a different context (open-loop observation) from a different approach to stability (Lyapunov's direct method), and for a more restricted (linear in the control) class of plants. As regards to Krener and Isidori's (1983) open-loop observer approach, its restrictive condition to have linear output error dynamics has been circumvented by our approach, where the output error dynamics is not required to be strictly linear, as is the case in high-gain observer designs. If the plant (Eq. 1) is MIMO and has Hammouri and Busawon's (1993) particular (and for process control, very restrictive) completely-linearizable form, both our approach and their approach yield the same output-feedback controller and different tuning procedures. In their approach, the control matrix is fixed, and the observer matrix is obtained from the solution of a parametrized Lyapunov-like matrix equation, as usual in high-gain observer designs. In the present approach, the calculation of the parametrized controller and observer gains is straightforward and there is higher flexibility, because the designer can vary the controller and observer speediness/retardation parameters (s_c and s_o) as well as the entries of the reference gains (K_{cr} and K_{or}). Also, the tuning procedure can be assisted with the ample repertoire of techniques for linear SISO control systems (D'azzo and Houpis, 1981; Morari and Zafiriou, 1989) to be chosen according to the specific plant and control objective.

Our conclusions on stability have been local in the sense that results are valid in some neighborhood (of undefined size and shape, not necessarily small) of the nominal steady state. At present, the global nonlinear geometric control only offers results for the state-feedback problem (Sussmann and Kokotovic, 1991). However, in applied process control, a global approach may neither be justified nor required (La Salle and Lefschetz, 1961), and the inherent complexity of a suitable nonlocal approach should apply to the operative part but not to the conceptual one. In principle, a numerically oriented Lyapunov approach with an optimization technique could be used to delimit the domain of attraction and its relationship with the state and output dynamics, the control action, etc., and the consideration of this goes beyond the scope of the present work.

Control of a Continuous Polymerization Reactor

Consider a stirred tank reactor where a monomer is polymerized via free radicals generated by initiator decomposition. The reaction is strongly exothermic and is accompanied by density change (contraction), significant viscosity increase, and heat-exchange capability decrease. The produced heat is removed by means of a cooling jacket. The reactor is represented in Figure 1. The states of the reactor are I (molar concentration of initiator), m (dimensionless molar concentration of monomer referred to pure monomer), and T (temperature). The reactor model is given by (Alvarez et al., 1990)

$$\dot{I} = -r_I(I,T) + \epsilon r_p(I,m,T)I + \frac{w_I - q_e I}{V}$$

$$\dot{m} = -(1-\epsilon m)r_p(I,m,T) + \frac{q_e}{V}(m_e - m)$$

$$\dot{T} = \beta(m)r_p(I,m,T) + \frac{(1-\epsilon m_e)}{(1-\epsilon m)}\frac{q_e(T_e - T)}{V} - \gamma(m,T)(T - T_c)$$

where V is the volume of the monomer-polymer mixture, q_e is the volumetric flow rate of monomer feed, w_I is the molar feed rate of initiator, m_e and T_e are the feed monomer concentration and temperature, respectively, T_c is the temperature of the cooling jacket, and $0 < \epsilon < 1$ is the contraction factor due to the difference in monomer and polymer densities. r_I is the rate of initiator decomposition, r_p is the polymerization rate, γ is proportional to the vessel-jacket heat-transfer coefficient which in turn depends on the viscosity of the monomer-polymer mixture, and β is a density-dependent adiabatic temperature rise determined by the heat of the reaction of the propagation step. r_I, r_p, γ, and β are smooth nonlinear functions. Due to the presence of the gel-effect, r_p and γ exhibit autoacceleration and significant viscosity increase with monomer conversion, respectively, which in turn may induce steady-state multiplicity (Henderson, 1987; Alvarez et al., 1990).

Let $x = (I, m, T)^T$ denote the reactor state. The measured outputs are: $y_1 = m$ (monomer concentration) and $y_2 = T$ (temperature). There are two control inputs: $u_1 = w_I$ (initiator feed rate in case I) or q_e (monomer feed rate in case II), and $u_2 = T_c$ (coolant temperature). In other words, we

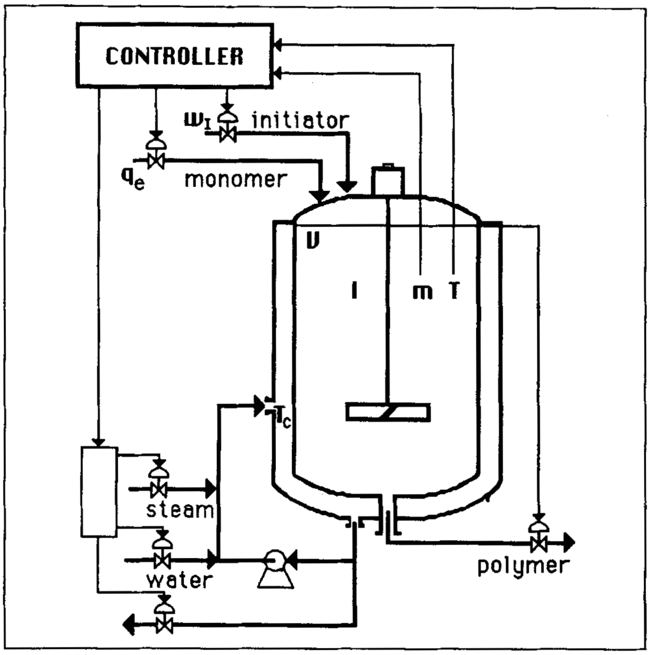

Figure 1. Polymerization reactor and its control scheme.

will study two control problems. In the notation of the second through fourth sections, the reactor control system is represented as follows

$$\dot{x} = f(x,u), \quad y = h(x) \tag{28}$$

where

$$x_1 = I, \quad x_2 = m, \quad x_3 = T, \quad y_1 = x_2, \quad y_2 = x_3, \quad u_2 = T_c$$

and

$$f(x,u) = [f_1(x,u_1), f_2(x), f_3(x,u_2)]^T, \quad u_1 = w_I (\text{case I})$$

$$f(x,u) = [f_1(x,u_1), f_2(x,u_1), f_3(x,u_1,u_2)]^T,$$

$$u_1 = q_e \ (\text{case II})$$

The reactor must operate at a given (possibly open-loop unstable) nominal state $\bar{x} = (\bar{I}, \bar{m}, \bar{T})$ associated to the nominal inputs $\bar{w}_I$, $\bar{q}_e$, and $\bar{T}_c$. For the reactor output feedback problems, emphasis will be placed on solvability conditions with physical significance, resorting to numerical simulations just for corroborative and illustrative purposes.

Solvability of the reactor output-feedback problem: Case I. $u = [w_I, T_e]^T$

Since $\gamma(x) \neq 0$ and $\partial r_p(x)/\partial I > 0$, the conditions of theorem 1 are met with the maps (Alvarez et al., 1990)

$$\phi(x) = [x_1, f_2(x), x_3]^T,$$

$$\varphi(x,u) = [L_{f(x,u)} f_2(x), f_3(x,u_2)]^T, \quad (\kappa_1, \kappa_2) = (2,1)$$

implying that the state-feedback problem is solvable with complete linearization. From theorem 2, the output-feedback problem is solved by the controller (Eq. 27) with s_o sufficiently large, and the following feedback and observer maps

$$\mu(\chi, K_{cr}, s_c) = \varphi^{-1}\{\chi, K_c(K_{cr}, s_c)[\phi(\chi) - \phi(\bar{x})]\},$$

$$K_c(K_{cr}, s_c) = \begin{bmatrix} s_c^2 k_{11}^{cr} & s_c k_{12}^{cr} & 0 \\ 0 & 0 & s_c k_{21}^{cr} \end{bmatrix},$$

$$G[\chi, K_o(K_{or}, s_o)] = [\phi_x^{-1}(\chi)]K_o,$$

$$K_o(K_{or}, s_o) = \begin{bmatrix} s_o k_{11}^{or} & 0 \\ s_o^2 k_{21}^{or} & 0 \\ 0 & s_o k_{12}^{or} \end{bmatrix}$$

where the reference gains are the coefficients of the LNPA output dynamics (Eq. 26)

$$\ddot{\eta}_1 - k_{12}^{cr}\dot{\eta}_1 - k_{11}^{cr}\eta_1 = 0, \quad \dot{\eta}_2 - k_{21}^{cr}\eta_2 = 0 (\text{controller}) \tag{29a}$$

$$\ddot{\upsilon}_1 + k_{11}^{or}\dot{\upsilon}_1 + k_{21}^{or}\upsilon_1 = 0, \quad \dot{\upsilon}_2 + k_{12}^{or}\upsilon_2 = 0 (\text{observer}) \tag{29b}$$

In terms of reference (1.83%)-settling times (τ_m^c, τ_T^c, τ_m^o, $\tau_T^o > 0$) and damping factors (ξ_m^c, $\xi_m^o > 0$), the reference gains are given by

$$k_{11}^{cr} = -16/(\xi_m^c \tau_m^c)^2, \quad k_{12}^{cr} = -8/\tau_m^c, \quad k_{21}^{cr} = -4/\tau_T^c \tag{30a}$$

$$k_{21}^{or} = 16/(\xi_m^o \tau_m^o)^2, \quad k_{11}^{or} = 8/\tau_m^o, \quad k_{12}^{or} = 4/\tau_T^o \tag{30b}$$

The solvabilities of the state-feedback and output-feedback reactor problems follow from two physical conditions that are met by any free-radical homopolymerization reactor: (i) $\gamma(m, T) > 0$ (the reactor is not adiabatic); and (ii) $\partial r_p(I, m, T)/\partial I > 0$ (the polymerization rate increases with the initiator concentration).

Case II: $u = [q_e, T_c]^T$

From $m \neq m_e$ and $\gamma(m, T) > 0$, it follows that conditions (i) to (iv) of theorem 1 are met with the following maps

$$\phi_I(x) = [x_2, x_3]^T, \quad \phi_{II}(x) = x_1, \quad (\kappa_1, \kappa_2) = (1,1)$$

The associated zero-dynamics (Eq. 7) is the following one-dimensional initiator dynamics ($x_1 = I$)

$$\dot{I} = -r_1(I, \bar{T}) - I r_p(I, \bar{m}, \bar{T})[(1 - \epsilon m_e)/(m_e - \bar{m})]$$

$$+ (w_I/V) := \theta(I)$$

which is A-stable because

$$d\theta(I)/dI = -\{(\partial r_I/\partial I)$$

$$+ [r_p + I(\partial r_p/\partial I)][(1 - \epsilon m_e)/(m_e - \bar{m})] < 0$$

implying that condition (v) of theorem 1 is met. From theorem 2, the output-feedback problem is solved by the controller (Eq. 27) with s_o and s_c sufficiently large and small, respectively, and the following feedback and observer maps

$$\mu(\chi, K_{cr}, s_c) = \varphi^{-1}\{\chi, K_c(K_{cr}, s_c)[\phi_1(\chi) - \phi_I(\bar{x})]\},$$

$$K_c(K_{cr}, s_c) = \begin{bmatrix} s_c k_{11}^{cr} & 0 \\ 0 & s_c k_{21}^{cr} \end{bmatrix},$$

$$G[K_o(K_{or}, s_o)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} K_o(K_{or}, s_o),$$

$$K_o(K_{or}, s_o) = \begin{bmatrix} s_o k_{11}^{or} & 0 \\ 0 & s_o k_{12}^{or} \end{bmatrix},$$

where the reference gains are the coefficients of the stable LNPA dynamics (Eq. 26)

$$\dot{\eta}_1 - k_{11}^{cr}\eta_1 = 0, \quad \dot{\eta}_2 - k_{21}^{cr}\eta_2 = 0 \quad \text{(controller)} \quad (31a)$$

$$\dot{v}_1 + k_{11}^{or}v_1 = 0, \quad \dot{v}_2 + k_{12}^{or}v_2 = 0 \quad \text{(observer)} \quad (31b)$$

In terms of reference settling times, the reference gains are given by

$$k_{11}^{cr} = -4/\tau_m^c, \quad k_{21}^{cr} = -4/\tau_T^c, \quad k_{11}^{or} = 4/\tau_m^o, \quad k_{12}^{or} = 4/\tau_T^o \quad (32)$$

The solvabilities of the state-feedback and output-feedback problems follow from four physical conditions that are met by any free-radical homopolymerization reactor: (i) $m_e - m > 0$ (the reactor monomer concentration is smaller than the one of the feedstream because $r_I > 0$ and $r_p > 0$); (ii) $\gamma(m, T) > 0$ (the reactor is not adiabatic); (iii) $\partial r_I(I,T)/\partial I > 0$ (the initiator decomposition rate increases with the initiator concentration); and (iv) $\partial r_p(I, m, T)/\partial I > 0$ (the polymerization rate increases with the monomer concentration).

Corroboration and illustration with numerical simulations

To subject the control schemes to a severe test, let us consider an extreme case design of a practical situation: the reactor operates at a high "fraction of solids" (mass fraction of polymeric material), the monomer has high reactivity with intense gel-effect, and the reactor must operate at a nominal state $\bar{x}$ that is open-loop unstable. The monomer is methyl methacrylate, and the initiator is AIBN. The model functionalities (r_I, r_P, β and γ), their parameters, and the operation condition are from Alvarez et al. (1994). The reactor holds a volume V of 2,000 L, and processes a nominal feed flow rate $\bar{q}_e$ of 40 L/min of pure monomer ($m_e = 1$) at $T_e = 300$ K. For nominal values of initiator feed rate $\bar{w}_I = 0.08$ gmol/min and coolant temperature $\bar{T}_c = 300$ K, the reactor has three steady states: $\bar{x} = [\bar{I}$ (gmol/L), $\bar{m}$, $\bar{T}$ (K); S(stable) or U(unstable)$]^T$ = (0.002014, 0.9933, 314.62; S), = (0.001831, 0.5807, 349.66; U), (0.0003241, 0.4331, 377.06; S). The first and third steady states are regarded as undesired extinction and ignition open-loop stable operations.

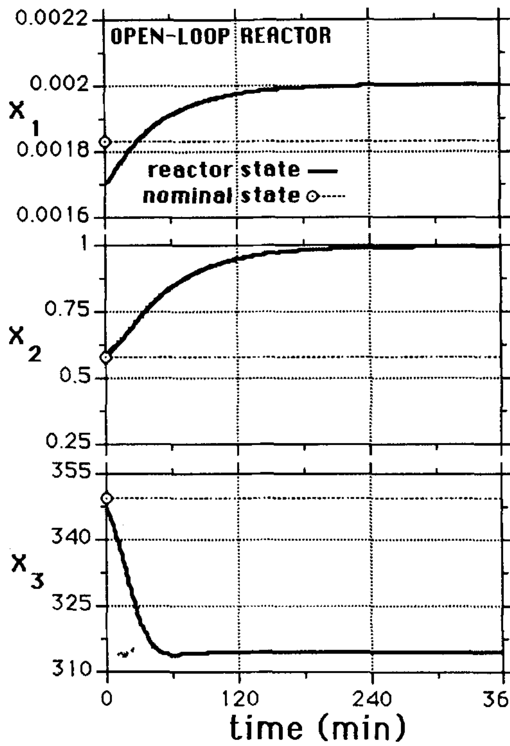

Figure 2. State responses of the open-loop reactor.

Figure 2 shows the open-loop response of the reactor when its initial state x_o is deviated from the nominal steady state $\bar{x}$: $x_o = (0.0017,\ 0.595,\ 347)^T \neq \bar{x}$. In this case, the reactor reaches the undesired extinction steady state.

For case I, the reference gains for the controller and observer designs are the ones used by Alvarez et al. (1990) in their state-feedback controller for the same polymerization reactor: $\xi_m^c = \xi_m^o = 0.71$, $\tau_m^c = \tau_m^o = 125$ min, $\tau_T^c = \tau_T^o = 50$ min, which were set to obtain the fastest state-feedback control response without saturation of controls and to have tolerance to modeling errors. Following theorem 2, let us fix the controller parameter at $s_c = 1$ and tune the observer parameter $s_o > 1$. At $s_o \approx 6.4$, the output-feedback controller stabilizes the reactor. In Figure 3, the state and control responses are shown for: $s_o = 7$ (i.e., the observer reference dynamics are seven times faster than the one of the controller design), the initial reactor state deviation is equal to the one of the open-loop run of Figure 2, and the initial state detector is $\chi_o = (0.0018, 0.575, 350) \neq x_o$. As expected from the severe condition design, drastic changes in the initiator feed rate and the coolant temperature are required to stabilize the reactor. The settling time of the controlled reactor is about three times faster that the one reported earlier by Adebakun and Shork (1989a) in their simulation study of the nonlinear control of a similar reactor. In practice, the initiator addition policy of Figure 2 can be implemented, but realizing the coolant temperature changes with a standard heating/cooling system could be problematic. However, the control action can be made arbitrarily smooth by choosing the control parameter s_c sufficiently small (i.e., by relaxing the requirement on the overall speed of response of the closed-loop reactor), and/or by increasing the temperature control settling time τ_T^c of the reference design.

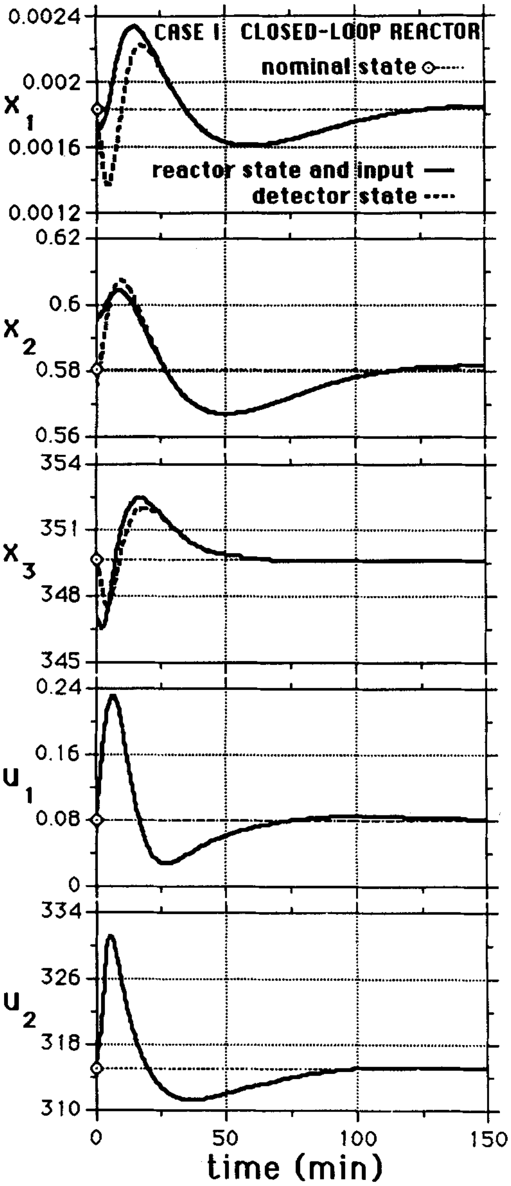

Figure 3. Closed-loop reactor with the output-feedback control of case I.

In case II, the presence of the initiator zero-dynamics inexorably limits the attainable closed loop response of the reactor, and therefore, to prevent unduly intense or excessive control action, the LNPA control design must not be chosen to be much faster than 150 min, which is an estimate of the settling time of the initiator zero-dynamics: $150 \approx 4[r_I(\overline{T})+\overline{q}_e/V]^{-1}$. Thus, the reference temperature settling time τ_T^c of case I is increased from 50 to 125 min. Following theorem 2, let us fix the controller parameter at $s_c=1$ and tune the observer parameter $s_o>1$. At $s_o \approx 3$, the output-feedback controller stabilizes the reactor. In Figure 4, the state and control responses are shown for $s_o=5$ (i.e., the observer reference dynamics is five times faster than the one of the controller design), and the reactor and detector initial conditions are those of case I. As expected, the control action is less drastic than in case I because: the reference temperature control settling time τ_T^c is larger, the two controllers are fully dedicated to steer a two-dimensional part (which includes the unstable part) of the three-dimensional reactor, and the zero-dynamics takes care of steering the remaining one-dimensional part of the reactor.

The preceding two control schemes have been designed and tested for nominal stability. As mentioned before, some robustness features have been already incorporated into the nominal output-feedback design: the parameter-independent condition number (i.e., the modulus ratio of the fastest to the slowest pole) of the control and observer designs have been fixed at one, and the tolerance of the state-feedback design to modeling errors has been tested earlier in Alvarez et al. (1990). From the control theory (Byrnes et al., 1991), we know that tolerance to modeling errors is favored by low orders of state-feedback linearization. This means that the reactor control scheme with complete linearization should be more sensitive to modeling errors than the scheme with partial lin-

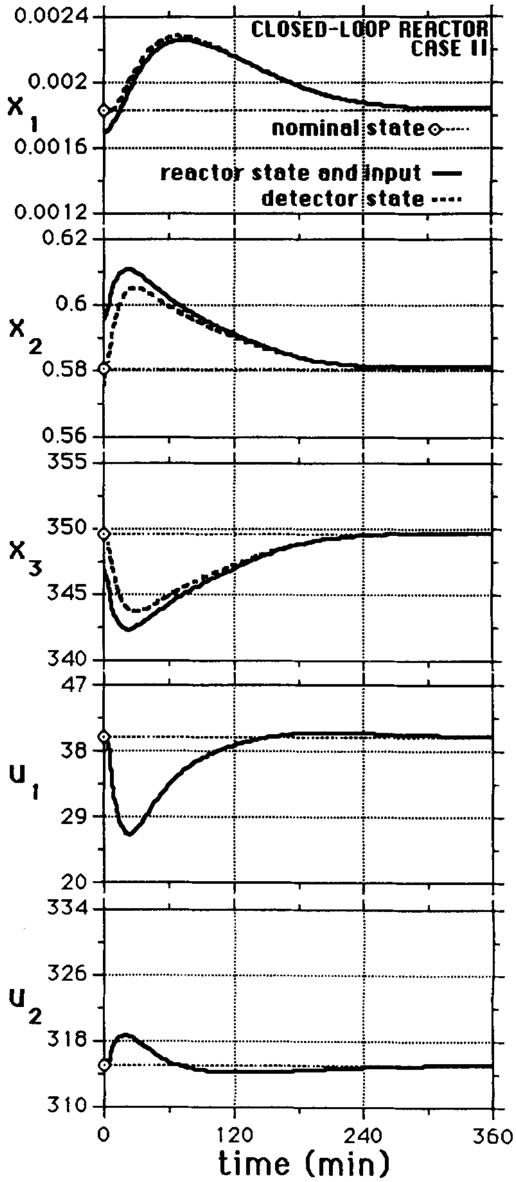

Figure 4. Closed-loop reactor with the output-feedback control of case II.

earization. In view of this, let us consider a test for modeling error for the worst situation: case I. The output-feedback control of case I (Figure 2) was re-run with the following erroneous parameters for the controller: E_p/R (E_p: activation energy of the propagation step) = 2,190 instead of 2,080 cal, a_h (dimensionless proportionality constant of the heat-exchange function) = 0.612 instead of 0.74, f_I (efficiency factor in the generation of free-radicals by the initiator decomposition) = 0.522 instead of 0.58, and $\epsilon = 0$ (ignoring the correction of mass and heat concentrations by the contraction of the reacting mixture). These errors signify a $\approx 36\%$ overestimation of the rate of the heat-producing propagation reaction, a $\approx 18\%$ underestimation in the capability of heat removal, a 14% underestimation of the initiator efficiency, and a $\approx 13\%$ underestimation of the density of the monomer-polymer mixture. Both the operating condition and the modeling errors are more drastic than the ones used by Adebakun and Shork (1989a,b) in their simulation and experimental studies on the nonlinear control problem of a similar reactor. The corresponding response of the closed-loop reactor is shown by the discontinuous curves of Figure 5. The controller manages to avoid extinction or ignition regimes, and there are state and control steady-state offsets, which is a feature that is consistent with an assessment derived from the LNPA output design when visualized within a conventional industrial-type framework: without integral action, the "proportional" nominal control design and the modeling errors produce a steady-state offset. Following the same conventional framework and its adaptation for the geometric state-feedback control design (Alvarez et al., 1990), let us redesign the "proportional" nonlinear controller by incorporating integral action

$$\mu(\chi, K_{cr}, s_c) = \varphi^{-1}\{\chi, K_c(K_{cr}, s_c)[\phi(\chi) - \phi(\bar{\chi})] + \chi_e\},$$

$$\dot{\chi}_e = K_I(y - \bar{y})$$

$$K_I(K_I^r, s_c) = \mathrm{diag}[s_c^3 k_1^{Ir}, s_c^2 k_2^{Ir}], \quad K_I^r = \mathrm{diag}[k_1^{Ir}, k_2^{Ir}]$$

where k_1^{Ir} and k_2^{Ir} are reference gains for the integral action in the monomer and temperature loops, respectively. In this case, the controller LNPA output dynamics design Eq. 29a becomes

$$\dddot{\eta}_1 - k_{12}^{cr}\ddot{\eta}_1 - k_{11}^{cr}\dot{\eta}_1 - k_1^{Ir}v_1 = 0, \quad \ddot{\eta}_2 - k_{21}^{cr}\dot{\eta}_2 - k_2^{Ir}\eta_2 = 0$$

To realize the integral action, let us add two real poles $\lambda_m^{cI} = -4s_c/\tau_m^{cI}$ (monomer) and $\lambda_T^{cI} = -4s_c/\tau_T^{cI}$ (temperature) to the reference "proportional" design. In this case, the gains of the preceding reference output dynamics are given by

$$k_{12}^{cr} = -8/\tau_m^c - 4/\tau_m^{cI}, \quad k_{11}^{cr} = -16/(\xi_m^c\tau_m^c)^2 - 32/(\tau_m^c\tau_m^{cI}),$$

$$k_1^{Ir} = -64/\left[(\xi_m^c\tau_m^c)^2\tau_m^{cI}\right]$$

$$k_{21}^{cr} = -4/\tau_T^c - 4/\tau_T^{cI} \quad k_2^{Ir} = -16/(\tau_T^c\tau_T^{cI})$$

Now, the monomer and temperature LNPA output dynamics are of orders three and two, respectively. The parameters $\xi_m^c = \xi_m^o = 0.71$, $\tau_m^c = \tau_m^o = 125$ min, $\tau_T^c = \tau_T^o = 50$ min, $s_c = 1$ are fixed at the values of the proportional design of case I, and the observer speediness factor is set at $s_o = 10$. To begin, a small integral action was considered by choosing long settling

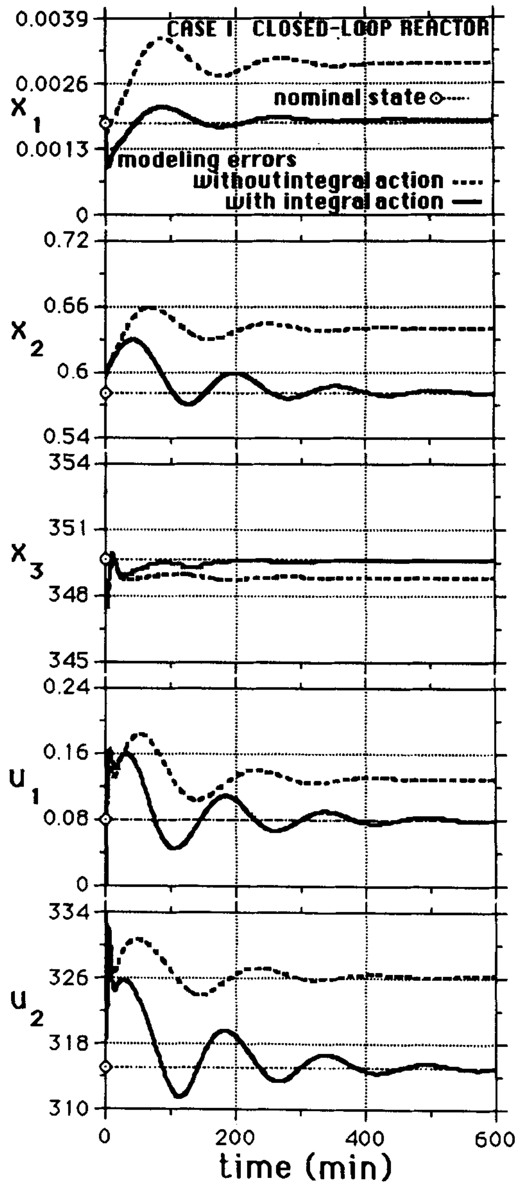

Figure 5. Closed-loop reactor with the output-feedback control of case I and modeling errors.

times: $\tau_m^{cI} = 1{,}500$, $\tau_T^{cI} = 500$ min, taking the reactor to its nominal state-input values in 1,200 min. After a few trials, the response shown by the continuous lines of Figure 5 was obtained with $\tau_m^{cI} = 200$ and $\tau_T^{cI} = 100$ min. As expected, this response is slower and more oscillatory than the one of the nominal controller without modeling errors and integral action (Figure 2), and no further effort was made to obtain a better response by retuning the controller; a more detailed tuning study should achieve this. On the other hand, the closed-loop response of Figure 5 is faster than the ones reported earlier in simulation and experimental studies (Adebakun and Shork, 1989a,b) on a similar reactor.

Conclusion

For MIMO nonlinear plants whose state-feedback problem is solvable with complete or partial output-linearization, the output-feedback problem has been solved by combining the state-feedback controller with a suitable closed-loop nonlinear state detector. Sufficient conditions for nominal closed-loop stability were given, the interplay between the controller and observer designs was identified, and a systematic and simple procedure to tune the controller-observer gains was obtained. Both the stability conditions and the tuning of gains were interpreted within a linear conventional-type framework that should be appealing to process control engineers, as compared with the more abstract and mathematically oriented construction and tuning procedures of the high-gain observer designs. *A priori*, that is, before simulation and testing, the solvabilities of two output-feedback problems in a free-radical homopolymerization reactor were characterized with rigorous analytic conditions bearing physical meaning, and their output-feedback controllers were designed. In a second step, these findings were tested and corroborated with numerical simulations. Thirdly, with an *ad hoc* incorporation of integral action into the control design, the controller was shown to handle modeling errors. This evidence, drawn at the simulation stage level, can be regarded as a point of departure for a study on the problem of robust stability.

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Appendix A: Matrices and Maps

Matrices

bd M: = block-diagonal matrix M

$$\Gamma_{(n\times n)} = bd[\Gamma_1,\ldots,\Gamma_m],\quad \Pi_{(n\times m)} = bd[\pi_1,\ldots,\pi_m],$$

$$\Delta_{(m\times n)} = bd[\delta_1,\ldots,\delta_m]$$

$$\Gamma_{i_{(\kappa_i\times\kappa_i)}} = \begin{bmatrix} 010\cdot\cdot 00 \\ 001\cdot\cdot 00 \\ ::: \;\; :: \\ 000\cdot\cdot 01 \\ 000\cdot\cdot 00 \end{bmatrix},\quad \pi_{i_{(\kappa_i\times 1)}} = \begin{bmatrix} 0\\0\\:\\0\\1 \end{bmatrix},\quad \delta_{i_{(1\times\kappa_i)}} = [1,0,\cdot\cdot,0,0]$$

$$A_{c(n\times n)} = \Gamma + \Pi K_c,\quad A_{o(n\times n)} = \Gamma - K_o\Delta$$

Sets $\mathcal{K}_c$ and $\mathcal{K}_o$

$$\mathcal{K}_c = \{K_{c(m\times n)} = bd[k_1^c,\ldots,k_m^c] \mid k_i^c = [k_{i1}^c,\ldots,k_{i\kappa_i}^c] \in \mathcal{k}_i^c\},$$

$$\mathcal{k}_i^c = \{k_i^c \mid \text{Eq. 5 is stable}\}$$

$$\mathcal{K}_o = \left\{K_{o(n\times m)} = bd[k_1^o,\ldots,k_m^o] \mid k_i^o = [k_{1i}^o,\ldots,k_{\kappa_i i}^o]^T \in \mathcal{k}_i^o\right\},$$

$$\mathcal{k}_i^o = \{k_i^o \mid \text{Eq. 12 is stable}\}$$

Nonlinear maps

$$\alpha(z_I, z_{II}) = \phi^{-1}[z + \phi(\bar{x})],$$

$$\nu(z_I, z_{II}) = \varphi^{-1}[\alpha(z_I, z_{II}), K_c z_I]$$

$$w(z_I, z_{II}, \upsilon) = \{\phi_{IIx}(x) f[x, \varphi^{-1}(x,\upsilon)]\}_{x=\phi^{-1}[z+\phi(\bar{x})]}$$

$$w^*(e_I, e_{II}, \zeta_I, \zeta_{II}) = w[\zeta_I, \zeta_{II}, \nu(\zeta_I, \zeta_{II})] - w[\zeta_I - e_I, \zeta_{II} - e_{II}, \nu(\zeta_I, \zeta_{II})]$$

$$\omega(z_{II}) = w[0, z_{II}, \nu(0, z_{II}), \overline{\omega}(e_{II})] = -\omega(-e_{II})$$

$$\theta_I(e_I) = -K_o\Delta e_I,\ \theta_{II}(\zeta_I, \zeta_{II}) = w[\zeta_I, \zeta_{II}, \nu(\zeta_I, \zeta_{II})] - w[0, \zeta_{II}, \nu(0, \zeta_{II})]$$

$$q_I(e_I, e_{II}, \zeta_I, \zeta_{II}) = \varphi[\alpha(\zeta_I, \zeta_{II}), \nu(\zeta_I, \zeta_{II})] - \varphi[\alpha(z_I - e_I, z_{II} - e_{II}), \nu(\zeta_I, \zeta_{II})]$$

$$q_{II}(e_I, e_{II}, \zeta_I, \zeta_{II}) = w^*(e_I, e_{II}, \zeta_I, \zeta_{II}) - w^*(0, e_{II}, 0, 0)$$

Appendix B: Proof of Theorem 1

From conditions (i) and (ii) of theorem 1, the state-control coordinate change

$$z = \phi(x) - \phi(\bar{x}),\quad \upsilon = \varphi(x,u) \tag{B1}$$

takes the plant into the following normal form (Isidori, 1989)

$$\dot{z}_I = \Gamma z_I + \Pi\upsilon,\quad \eta = \Delta z_I;\quad \dim z_I = \kappa \tag{B2a}$$

$$\dot{z}_{II} = w(z_I, z_{II}, \upsilon);\quad \dim z_{II} = n - \kappa \tag{B2b}$$

where Γ, Π, Δ and w are defined in Appendix A, and υ is regarded as a new control input. Equation B2b with $z_I = 0$ and $\upsilon = 0$ is the E-stable zero-dynamics (Eq. 7) in z-coordinates. From the block-diagonality and the controllability of the pair (Γ, Π), there is a control gain matrix K_c (defined earlier) such that the linear controller

$$\upsilon = K_c z_I \tag{B3}$$

yields an A-stable closed-loop system (A_c is defined in Appendix A),

$$\dot{z}_I = A_c z_I,\quad \eta = \Delta z_I \tag{B4a}$$

$$\dot{z}_{II} = w(z_I, z_{II}, K_c z_I) \tag{B4b}$$

with the output dynamics (Eq. 5) yielding the two properties of definition 1. Recall the coordinate change (Eq. B1), write the controller (Eq. B3) in (x, u)-coordinates, and obtain the nonlinear controller (Eq. 8). **QED**

Appendix C: Proofs of Lemmas 1a and 1b

Let us recall Gronwall's lemma (see Vidyasagar, 1978): suppose $\sigma(t)$, $t \in [t_o, \infty)$, is a continuous function, and a_σ, $b_\sigma \geq 0$ are given constants. Then,

$$\sigma(t) \leq a_\sigma + b_\sigma\int_0^t \sigma(\tau)d\tau \Rightarrow \sigma(t) \leq a_\sigma \exp(b_\sigma t) \tag{C1}$$

As a preliminary step for the proof of lemmas 1a and 1b, let us introduce the following proposition to establish an integral inequality set for the bounds of the four motions of the closed-loop plant (Eq. 18).

Proposition 1

In some neighborhood of the origin, the motions of the closed-loop plant (Eq. 17 or 18) are bounded as follows

$$\|e_I(t)\| \leq a_o\|e_{Io}\|\exp[-(\lambda_o - a_o L_1)t] + a_o L_2\int_0^t \times\{\exp[-(\lambda_o - a_o L_1)(t-\tau)]\}\|e_{II}(\tau)\|d\tau \tag{C2a}$$

$$\|\zeta_I(t)\| \leq a_c\|\zeta_{Io}\|\exp(-\lambda_c t) + a_c k_o\int_0^t\{\exp[-\lambda_c(t-\tau)]\}\|e_I(\tau)\|d\tau, \tag{C2b}$$

$$\|\zeta_{II}(t)\| \leq a_z\|\zeta_{IIo}\|\exp(-\lambda_z t) + b_z L_3\int_0^t\{\exp[-\lambda_z(t-\tau)]\}\|\zeta_I(\tau)\|d\tau, \tag{C2c}$$

$$\|e_{II}(t)\| \leq a_z\|e_{Io}\|\exp(-\lambda_z t) + b_z\int_0^t\{\exp[-\lambda_z(t-\tau)]\} \times[L_4\|e_I(\tau)\| + L_5\|\zeta_I(\tau)\| + L_6\|\zeta_{II}(\tau)\|]d\tau. \blacklozenge \tag{C2d}$$

Proof

Keeping the set $E \times Z$ and the gains K_c and K_o fixed for the present, take the time-integration of Eqs. 18a and 18b, and obtain

$$e_I(t) = [\exp(A_o t)]e_I(0) + \int_0^t \{\exp[A_o(t-\tau)]\}\Pi q_I[e_I(\tau), e_{II}(\tau), \zeta_I(\tau), \zeta_{II}(\tau)]d\tau \quad \text{(C3a)}$$

$$\zeta_I(t) = [\exp(A_c t)]\zeta_I(0) + \int_0^t \{\exp[A_c(t-\tau)]\}\theta_I[e_I(\tau)]d\tau \quad \text{(C3b)}$$

Take norms in Eq. C3b, substitute In. 23b, and obtain In. C2b. Take norms in Eq. C3a, substitute In. Eq. 23a, multiply the resulting inequality by $\exp(\lambda_o t)$, let $\|e(t)\| \exp(\lambda_o t) := \sigma(t)$, apply Gronwall's lemma (C1), replace $\sigma(t)$ by $\|e(t)\| \exp(\lambda_o t)$, and obtain the inequality (Eq. C2a).

From Lyapunov's converse theorem (Khalil, 1992), there is a Lyapunov function $V(z_{II})$, and four positive constants, $c_i > 0$, such that the following inequalities

$$c_1\|z_{II}\| \le V(z_{II}) \le c_2\|z_{II}\|, \quad \|(\partial V/\partial z_{II})\omega(z_{II})\| \le -c_3\|z_{II}\|, \quad \|(\partial V/\partial z_{II})\| \le -c_4 \quad \text{(C4)}$$

hold along the motion $z_{II}(t)$ of the zero dynamics (Eq. 20). Regard $V(z_{II})$ as a candidate Lyapunov function for the "disturbed" closed-loop subsystem (Eq. 18c), take the time-derivative of $V(\zeta_{II})$ along the motion $\zeta_{II}(t)$ of the system (Eq. 18c), substitute the preceding inequality set and the inequality (Eq. 23c), and obtain that

$$\dot{V} \le -(c_3/c_1)V + c_4 L_3 \|\zeta_I\|$$

The integration of this inequality, followed by the substitution of the inequalities $\|\zeta_{II}(t)\| \le V(t)/c_1$ and $V(0) \le c_2\|\zeta_{II}(0)\|$ (obtained from the expression of In. C4), yields

$$\|\zeta_{II}(t)\| \le (c_2/c_1)\{\exp[-(c_3/c_1)t]\}\|\zeta_{IIo}\| + (c_4/c_1)L_3\int_0^t \{\exp[-(c_3/c_1)(t-\tau)]\}\|\zeta_I(\tau)\|d\tau \quad \text{(C5)}$$

If $\zeta_{II} = z_{II}$ and $L_3 = 0$, this inequality coincides with the inequality 21, and their comparison yields that: $a_z = c_2/c_1$, and $\lambda_z = c_3/c_1$. Thus, the third inequality of the inequality set C4 can be rewritten as follows

$$\|(\partial V/\partial z_{II})\| \le -b_z c_1, \quad b_z = c_4/c_1 \quad \text{(C6)}$$

With the preceding identification of constants in the inequality C5, one obtains the inequality C2c.

From the definition (Appendix A) of maps ω and $\overline{\omega}$, and from the stability of the zero-dynamics, the motion $e_{II}(t)$ of the subsystem (Eq. 18d) with $q_{II} = 0$ is bounded as follows: $\|e_{II}(t)\| \le a_z\|e_{II_o}\| \exp(-\lambda_z t)$. For the motion $e_{II}(t)$ of the same system (Eq. 18 with $q_{II} = 0$), there is a Lyapunov function that satisfies an inequality set of the form C4. With the same procedure employed in the proof of the inequality C2c, and recalling the Lipschitz inequality 23d, one obtains the inequality C2d. This proves proposition 1. **QED**

Proof of lemma 1a

If $\kappa = n$ ($e = e_I$ and $\zeta_I = e_I$), the inequality set C3 reduces to

$$\|e(t)\| \le a_o\|e_o\|\exp[-(\lambda_o - a_o L_1)t]$$

$$\|\zeta(t)\| \le a_c\|\zeta_o\|\exp(-\lambda_c t) + a_c k_o \int_0^t \{\exp[-\lambda_c(t-\tau)]\}\|e(\tau)\|\, d\tau$$

From the stability condition of lemma 1a, there is some neighborhood $E \times \mathcal{Z}$ where $\lambda_o - a_o L_1(E, \mathcal{Z}, K_c) > 0$, so that $e(t)$ and $\zeta(t)$ vanish asymptotically, or equivalently, the closed-loop plant (Eq. 19) is asymptotically stable. **QED**

Proof of lemma 1b

In matrix form, the inequality C2 can be rewritten as follows

$$r(t) \le A_1 S(t) r_o + \int_0^t A_2 S(t-\tau) r(\tau)\, d\tau \quad \text{(C7)}$$

where

$$r(t) = [\|e_I(t)\|, \|\zeta_I(t)\|, \|\zeta_{II}(t)\|, \|e_{II}(t)\|]^T,$$

$$S(t) = \exp\{\mathrm{diag}[-(\lambda_o - a_o L_1), -\lambda_c, -\lambda_z, -\lambda_z]t\}$$

$$A_1 = \mathrm{diag}[a_o, a_c, a_z, a_z],$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & a_o L_2 \\ a_c k_o & 0 & 0 & 0 \\ 0 & b_z L_3 & 0 & 0 \\ b_z L_4 & b_z L_5 & b_z L_6 & 0 \end{bmatrix}$$

Take the Laplace transform (LT) of the inequality C7 and obtain

$$\rho(s) \le A_1 \Sigma(s) r_o + A_2 \Sigma(s) \rho(s); \quad \rho(s) = LT[r(t)], \quad \Sigma(s) = LT[S(t)]$$

$$\Sigma(s) = \mathrm{diag}\{1/[s + (\lambda_o - a_o L_1)], \quad 1/(s + \lambda_c), \quad 1/(s + \lambda_z), 1/(s + \lambda_z)\}$$

multiply the last inequality by s, take the limit $s \to 0$, recall the final value theorem [i.e., $sF(s) \to f(\infty)$ as $s \to 0$], and obtain the following matrix inequality

$$Ar(\infty) \le 0, \quad r(\infty) \ge 0$$

$$r(\infty) = \begin{bmatrix} \|e_I(\infty)\| \\ \|\zeta_I(\infty)\| \\ \|\zeta_{II}(\infty)\| \\ \|e_{II}(\infty)\| \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 & -a_oL_2/(\lambda_o - a_oL_1) \\ -a_ck_o/\lambda_c & 1 & 0 & 0 \\ 0 & -b_zL_3/\lambda_z & 1 & 0 \\ -b_zL_4/\lambda_z & -b_zL_5/\lambda_z & -b_zL_6/\lambda_z & 1 \end{bmatrix}$$

$$\det A = 1 - [a_oL_2/(\lambda_o - a_oL_1)]\{b_zL_4/\lambda_z + a_ck_o/\lambda_c[b_zL_5/\lambda_z + (b_zL_3/\lambda_z)(b_zL_6/\lambda_z)]\}$$

From the two conditions of lemma 1b follows that, as $E \times \mathcal{Z} \hookrightarrow 0$

$$\lambda_o - a_oL_1(E, \mathcal{Z}, K_c) > 0, \det[A(E, \mathcal{Z}, K_c, K_o)] \neq 0$$

and therefore, these inequalities hold in some neighborhood $E \times \mathcal{Z}$. Hence, $r(\infty) = 0$, implying that the motion $r(t)$ of the inequality (Eq. C7) vanishes asymptotically, or equivalently, that the norms of the motions of the integral-form closed-loop dynamics (In. C2) vanish asymptotically. This means that the closed-loop (in original coordinates) dynamics (Eq. 15) is A-stable.

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Corrections

• The dimensions of the polarization parameter, *a*, are given as $cm^2/(v \bullet s)$ in the articles titled "Solute Retention in Electrochromatography by Electrically Induced Sorption" by S. R. Rudge, S. K. Basak, and M. R. Ladisch (May 1993, p. 797) and "Mechanistic Description and Experimental Studies of Electrochromatography of Proteins" by S. K. Basak and M. R. Ladisch (November 1995, p. 2499). The corrrect dimensions are $cm^2/(v \bullet min)$. We thank Dr. C. B. Chidambara Raj of the Centre for Research and Development, Southern Petro Chemical Industries Corp., Tamilnadu, India, for calling this to our attention.

• The title of an R&D note published on p. 2084 of the July 1996 issue should read "Improved Accuracy and Convergence of Discretized Population Balance of Litster et al." The name "Litster" was incorrectly published as "Lister."